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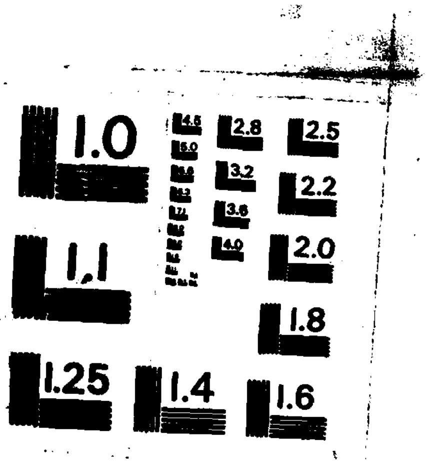

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Wave Vector Dependent Susceptibility of a Free Electron Gas in D Dimensions and the Singularity at $2k_F$

by

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ABSTRACT

The static susceptibility of a free electron gas in D dimensions at T = 0 is obtained by techniques of dimensional regularization. Our solutions for the susceptibility $\chi(k,D)$ are given in terms of the hypergeometric function. For any integer dimensions analytic expressions are possible. The high and low k series solutions are shown to be related by an analytic continuation if D is an odd integer, but not related if D is an even integer. The singularity at $2k_F$ is a branch point, whereupon the series solutions are absolutely convergent, yielding $\chi(k=2k_F,D) = (D-1)^{-1}$. The relationship of χkD has the appearance of a PVT diagram.

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I. INTRODUCTION

The wave vector k dependent susceptibility $\chi(k)$ is a basic physical quantity in many-body physics.[1] It enters into a variety of physical relationships, e.g., dispersion relations, scaling laws. For a free electron gas this quantity is exactly known in spatial dimensions D = 1, 2, and 3.[2-6] Especially interesting is its singular behavior at $k=2k_F$ due to what is known as the Pauli blocking, where k_F is the Fermi wave vector. This singularity is responsible for the Kohn anomaly in the phonon spectrum of a metal.[7] The singularity in the susceptibility at $2k_F$ is D dependent. In D = 1 the susceptibility has a logarithmic divergence. In D = 2 the susceptibility is finite but its slope is discontinuous. In D = 3 the slope has a logarithmic divergence. This trend suggests that the strength of the singularity becomes weaker with increasing dimensions. A precise knowledge of the D dependence would be of interest.

In addition to its own intrinsic interest, the susceptibility for a free electron gas is useful in other ways. If electrons now freely interact pairwise via the Coulomb force, the susceptibility for the interacting electron gas can always be written in the following form: $\chi^{int}(k) = \chi(k)/(1 + \Lambda_k \chi(k))$, where Λ_k is some function of the interaction.[8] If $\Lambda_k = v_k$, where v_k is the Fourier transform of the Coulomb potential, one gets the simple RPA theory. If $\Lambda_k = v_k(1 - G_k)$, where G_k is a local field term, one recovers the generalized RPA theory. Hence, the knowledge of the susceptibility for a free electron gas is essential to these RPA theories.

The above idea may be extended to the linear response theory of dynamical processes since $\chi(k) = \chi(k,\omega=0)$, where ω is the frequency. In dynamical theories, the knowledge of the static susceptibility is always pre-supposed.[1] For example, the relaxation function is normalized with respect to the static susceptibility. Moment sum rules are expressible in terms of the static susceptibility.[9]

More subtle is that the susceptibility may be defined by the Kubo scalar product (see sec. II). The Kubo scalar product is an inner product which realizes an abstract Hilbert space. In this realized space time-dependent quantum statistical problems are all definable.[10] Hence, the existence of the susceptibility plays a central role in the study of time evolution of dynamical variables.[11] In such a study there is the possibility that the relaxation function may assume a mean-field form in all spatial dimensions greater than a certain critical value.[12] This kind of dynamic anomaly is signaled by a deformation of the realized Hilbert space.[13] Furthermore, the critical dimension may take on a noninteger value. The physics of noninteger dimensions is of current theoretical interest. See e.g., fractals,[14] ε-expansions[15], kinetics of formation.[16]

The evaluation of the susceptibility for higher integer dimensions e.g., $D = 4$ may be carried out as was for $D = 1 - 3$ (see Secs. II and III). We shall use techniques of dimensional regularization developed in particle physics[17] to obtain a solution for the susceptibility which is valid for any D, integers and nonintegers. This solution $\chi(k,D)$ might be viewed as e.g., the PVT diagram of a homogeneous fluid. It traces a contour, which is a map of a continuous surface. This map is naturally divided into two regions (high and low k) by the D line at $k = 2k_F$. By moving alongside of this boundary line, one can examine the mathematical nature of the singularity at $2k_F$.

We find that $\chi(2k_F,D) = 1/(D-1)$, $D > 1$; $\chi'(2k_F,D) = -(D-2)/(D-1)(D-3)$, $D > 3$, etc. We also find that the susceptibility is of two families, D odd and D even. For D odd, the singularity at $2k_F$ is all logarithmic in origin. For D even (high k region) the singularity is of the square root. For D odd, the solution in one region is an analytic continuation into the other and $2k_F$ is a branch point. For D even, there are no such relationships and $k = 2k_F$ is a branch point only in the high k region.

II. STATIC SUSCEPTIBILITY

A free electron gas is described by the following Hamiltonian:

$$H = \Sigma_k \varepsilon_k c_k^+ c_k \tag{1}$$

where $\varepsilon_k = k^2/2m$, m is the mass of the electron, c_k^+ and c_k are, respectively, the fermion creation and annihilation operators at wave vector k. Our units are such that $\hbar = 1$. The longitudinal response to a weak static density-coupled perturbation is the static susceptibility given by the Kubo scalar product (K.S.P.)[9],

$$\chi(k) = (\rho_k, \rho_k^+) = \int_0^\beta d\lambda \, \langle e^{\lambda H} \rho_k e^{-\lambda H} \rho_k^+ \rangle \tag{2}$$

where β is the inverse temperature, the brackets $\langle \ldots \rangle$ denote an ensemble average, + denotes Hermitian conjugation, and ρ_k is the density fluctuation operator defined as

$$\rho_k = \sum_p c_p^+ c_{p+k}. \tag{3}$$

It is well known that the K.S.P. for a free electron gas may be given by

$$\chi(k) = 2\sum_p (f_p - f_{p+k})/(\varepsilon_{p+k} - \varepsilon_p) \tag{4}$$

where f_k is the Fermi function. Converting the sum into an integral we can rewrite (4) in spatial dimensions D as

$$\chi_D(k) = 4\,(2\pi)^{-D} \int d^D p \; f_p/(\varepsilon_{p+k} - \varepsilon_p). \tag{5}$$

At T = 0 the Fermi function is a step function, i.e., $f_k = \theta(k_F - k)$. Hence, for D small integers one can directly evaluate (5). For D = 1-3, the susceptibility is already known.[2-6] For comparison purposes, we shall list its normalized values $\tilde{\chi}_D(k) \equiv \chi_D(k)/\chi_D(0)$, expressing k in units of k_F:

$$\tilde{\chi}_1(k) = k^{-1} \ln|(2+k)/(2-k)| \tag{6a}$$

$$\tilde{\chi}_2(k) = 1 - \theta(k-2).(1 - 4/k^2)^{\frac{1}{2}} \tag{6b}$$

$$\tilde{\chi}_3(k) = \tfrac{1}{2}[1 + k^{-1}(1 - \tfrac{1}{4}k^2) \ln|(2+k)/(2-k)|]. \tag{6c}$$

III. EVALUATION OF THE SUSCEPTIBILITY FOR D = 4

The static susceptibility for D = 4 may be written down from (5):

$$\chi_4(k) = \frac{mk_F^2}{k} \int_0^1 dp\, p^3 \int_0^{2\pi} d\theta \frac{\sin^2\theta}{k + 2p\cos\theta} \quad . \tag{7}$$

We shall consider the angular integral first (denoted by Q). It may be converted into a contour integral on the unit circle by the substitution $u = e^{i\theta}$,

$$Q = -\frac{i}{8p} \oint du \frac{(u^2 - 1)^2}{u^2.(u - u_+).(u - u_-)} \tag{8}$$

where

$$u_\pm = k/2p \pm \left((k/2p)^2 - 1\right)^{\frac{1}{2}} .$$

The zeros of the denominator are 0, u_+ and u_-. The zeros u_+ and u_- may be real or complex depending on whether $|k/2p|$ is greater or less than 1.

(i) $|k/2p| > 1$

The conjugate zeros u_+ and u_- are real lying, respectively, outside and inside the unit circle. Hence, the zero u_+ does not contribute to the integral. The residues at 0 and u_- are, respectively, k/p and $-2((k/2p)^2 - 1)^{\frac{1}{2}}$. Together, $Q = \pi k/2p^2 . (1 - (1 - (2p/k)^2)^{\frac{1}{2}}$.

(ii) $|k/2p| < 1$

The conjugate zeros are now complex and lie on the contour of integration. If we take the Cauchy principal value, their contributions cancel each other exactly, leaving only the zero at the origin to contribute to the integral, giving $Q = \pi k/2p^2$.

Both may be combined to read as

$$Q = \pi k/2p^2 . \left[1 - \theta(k^2 - 4p^2).(1 - (2p/k)^2)^{\frac{1}{2}}\right] . \tag{9}$$

Using (9) in (7) we can now complete the radial part:

$$\tilde{\chi}_4(k) \equiv \chi_4(k)/\chi_4(0)$$

$$= 1 - k^2/6. \left[1 - \theta(k-2).(1 - 4/k^2)^{3/2}\right], \tag{10}$$

where $\chi_4(0) = mk_F^2/4\pi^2$. We observe that the susceptibility and its derivative are both finite and continuous at $k = 2k_F$.

For D even generally, the same idea may be used to evaluate the angular integral. There always is one pole of order D-2 at the origin. The conjugate poles u_+ and u_- behave in the same manner as described for D=4. For D odd, one cannot avail of this simplification and must resort to e.g., integration by parts. In any event, the evaluation of the susceptibility by this standard approach becomes very tedious as $D \to \infty$. Also one is limited to integer dimensions only. We shall, therefore, consider another approach i.e., dimensional regularization[17] which may allow us to obtain the susceptibility possibly more simply and, more important, in any dimensions.

IV. DIMENSIONAL REGULARIZATION

From (5) it is possible to express the zero temperature susceptibility for $D \geq 3$ as follows:[18]

$$\chi_D(k) = A \int_0^1 dp\, p^{D-1} \int_0^\pi d\theta\, (\sin\theta)^{D-2} . (\tfrac{1}{2}k + p\cos\theta)^{-1}$$

$$\equiv A\, I \tag{11}$$

where $A = A(k,D) = 2k_F^{\ D}\, S_{D-1}/k\varepsilon_F(2\pi)^D$, where $S_D = 2\Gamma(\tfrac{1}{2})^D/\Gamma(\tfrac{1}{2}D)$, and k is in units of k_F. To evaluate this double integral, we exchange the order of integration. For $k > 2$ the integrand is well behaved in the given interval of p. One can, therefore, expand it in powers of (2/k) and carry out the integration term by term. If, for $k < 2$, one attempts to expand it in powers of (k/2), one encounters a pole in the interval of p. To avoid this difficulty, we consider the following integral:

$$I_s = \int_0^\pi d\theta\, (\sin\theta)^{D-2} \int_0^1 dp\, p^{D-1} . (\tfrac{1}{2}k + p\cos\theta)^{2s-1} . \tag{12}$$

If we assume s is a positive integer, the new integrand is now well behaved in the interval of p for any k. Hence, one may expand it binomially and complete the integration term by term. Then one may possibly analytically continue I_s to obtain $I_0 \equiv I$. Clearly for $K > 2$ this process is unnecessary, hence it can be used as a direct test.

For any positive integer s, we obtain

$$I_s = \sum_{n=0}^{2s-1} \frac{(\tfrac{1}{2}k)^n\ \Gamma(2s)}{(D+2s-n-1) . \Gamma(2s-n) . \Gamma(n+1)} \int_0^\pi d\theta\, (\sin\theta)^{D-2} (\cos\theta)^{2s-n-1} . \tag{13}$$

Next the angular integration although cumbersome is straightforward. Terms of even n vanish. Letting $n \to 2n-1$, we get

$$I_s = \tfrac{1}{2} \sum_{n=1}^{s} (k/2)^{2n-1} \frac{\Gamma(2s) . \Gamma(\tfrac{1}{2}D-\tfrac{1}{2}) . \Gamma(s-n+\tfrac{1}{2})}{\Gamma(2n) . \Gamma(2s-2n+1) . \Gamma(s-n+\tfrac{1}{2}D+1)} . \tag{14}$$

The above expression is appropriate for the $k<2$ expansion. To obtain an expression more suitable for the $k>2$ expansion, we rewrite (14) in ascending powers of $(2/k)$ by letting $s-n \rightarrow n$,

$$I_s = \tfrac{1}{2}(k/2)^{2s-1} \sum_{n=0}^{s-1} (2/k)^{2n} \frac{\Gamma(2s).\Gamma(\tfrac{1}{2}D-\tfrac{1}{2}).\Gamma(n+\tfrac{1}{2})}{\Gamma(2s-2n).\,\Gamma(2n+1).\Gamma(n+\tfrac{1}{2}D+1)}. \qquad (15)$$

Both expressions (14 & 15) are clearly well defined for any finite positive integer s. There are s terms in each expansion. We now take advantage of the Gamma functions present in our expansions to perform analytic continuation. We first note that for any $r>0$,

$$\lim_{t\to -1} \Gamma(t+1) = t\Gamma(t) = (-1)..(-2)..(-r+1)(-r)\ \Gamma(-r).$$

Hence

$$\lim_{s\to 0} \frac{\Gamma(2s)}{\Gamma(2n).\Gamma(2s-2n+1)} = \frac{(-1)(-2)..(-2n+1).\Gamma(-2n+1)}{\Gamma(2n).\Gamma(-2n+1)} = -1$$

and

$$\lim_{s\to 0} \frac{\Gamma(2s)}{\Gamma(2s-2n).\Gamma(2n+1)} = \frac{(-1)(-2)..(-2n).\Gamma(-2n)}{\Gamma(-2n).\Gamma(2n+1)} = 1.$$

Using these results we get[19]

$$I\chi_0 = -\tfrac{1}{2}\Gamma(\tfrac{1}{2}D-\tfrac{1}{2}) \sum_{n=1}^{\infty} (k/2)^{2n-1}\ \Gamma(-n+\tfrac{1}{2})/\Gamma(-n+\tfrac{1}{2}D+1), \quad k<2 \qquad (16a)$$

$$= \tfrac{1}{2}\Gamma(\tfrac{1}{2}D-\tfrac{1}{2}) \sum_{n=0}^{\infty} (2/k)^{2n+1}\ \Gamma(n+\tfrac{1}{2})/\Gamma(n+\tfrac{1}{2}D+1), \quad K>2. \qquad (16b)$$

Finally, using the definition for A, $\chi(k{=}0,D)=(k_F/2\pi)^D.S_D/\varepsilon_F$,[20] where

$$S_D/S_{D-1} = \Gamma(\tfrac{1}{2}).\Gamma(\tfrac{1}{2}D-\tfrac{1}{2})/\Gamma(\tfrac{1}{2}D),$$

we get $\tilde{\chi}(k,D) \equiv \chi(k,D)/\chi(0,D)$

$$= -\tfrac{1}{2}\frac{\Gamma(\tfrac{1}{2}D)}{\Gamma(\tfrac{1}{2})} \sum_{n=1}^{\infty} (k/2)^{2n-2} \frac{\Gamma(-n+\tfrac{1}{2})}{\Gamma(-n+\tfrac{1}{2}D+1)}, \quad k<2 \qquad (17a)$$

$$= \tfrac{1}{2}\frac{\Gamma(\tfrac{1}{2}D)}{\Gamma(\tfrac{1}{2})} \sum_{n=0}^{\infty} (2/k)^{2n+2} \frac{\Gamma(n+\tfrac{1}{2})}{\Gamma(n+\tfrac{1}{2}D+1)}, \quad k>2. \qquad (17b)$$

Since the Gamma functions are well defined for any arguments other than zero or negative integers, our solutions (17a,b) are applicable to any value of D. Our series solutions agree term by term with the high and low k expansions of the susceptibility for D = 1-4 (eqs. 6 and 10) previously obtained by a conventional method.

V. APPLICATION OF THE HYPERGEOMETRIC FUNCTION

It is possible to express the susceptibility series (eqs. 17a,b) in terms of the hypergeometric series $F(a,b;c;t)$ defined as follows:[21]

$$F(a,b;c;t) = \sum_{n=0}^{\infty} (a_n\, b_n/c_n\, n!)\, t^n, \quad |t| < 1 \tag{18}$$

where $a_n = \Gamma(n+a)/\Gamma(a)$, etc., $c \neq 0, -1, -2, \ldots$ The advantages of having the susceptibility given in the hypergeometric function (h.g.f.) are evident. One can obtain analytic representations for integer dimensions. Properties of the h.g.f. may be used to study the behavior of the susceptibility at the singular point $2k_F$. The high and low k expansions may be related through an analytic continuation.

For this purpose, we introduce $z = (\tfrac{1}{2}k)^2$ and let $\tilde{\chi}_1(z,D) = \tilde{\chi}(z<1,D)$ and $\tilde{\chi}_2(z,D) = \tilde{\chi}(z>1,D)$. We shall consider $\tilde{\chi}_1(z,D)$ first. Using the identity

$$\Gamma(t-n) = (-1)^n\, \Gamma(t).\Gamma(-t+1)/\Gamma(-t+n+1)$$

in (17a) we obtain after some manipulations

$$\tilde{\chi}_1(z,D) = -\,\Gamma(\tfrac{1}{2})/D\Gamma(-\tfrac{1}{2}D)\,.\sum_{n=0}^{\infty} z^n\, \Gamma(n+1-\tfrac{1}{2}D)/\Gamma(n+3/2)$$

$$= F(1,1-\tfrac{1}{2}D;3/2;z)\,. \tag{19a}$$

Similarly, we obtain from (17b) with aid of (18)

$$\tilde{\chi}_2(z,D) = D^{-1}\, z^{-1}\, F(1,\tfrac{1}{2};1+\tfrac{1}{2}D;z^{-1})\,. \tag{19b}$$

Hence, together we have

$$\tilde{\chi}(z,D) = F(1,1-\tfrac{1}{2}D;3/2;z), \quad z < 1 \tag{20a}$$

$$= D^{-1}\, z^{-1}\, F(1,\tfrac{1}{2};1+\tfrac{1}{2}D;z^{-1}), \quad z > 1. \tag{20b}$$

We observe that for $D = 1$ the high and low sides of the susceptibility have the same parameters of the h.g.f.: $a = 1$, $b = \tfrac{1}{2}$, $c = 3/2$. For these values the h.g.f. has an analytic representation,

$$F(1,\tfrac{1}{2};3/2;t^2) = \tfrac{1}{2}t^{-1}\, \ln(1+t)/(1-t), \quad |t| < 1. \tag{21}$$

The resulting susceptibility is in exact agreement with the $D = 1$ result (eq. 6a).

To obtain analytic representations of the susceptibility for other integer dimensions, we study the h.g.f. Consider $F(1,1-\frac{1}{2}D;3/2;t)$ first. For D even, $b = 0,-1,-2,..$ and for D odd, $b = \frac{1}{2},-\frac{1}{2},-3/2,..$Hence, for D even, the h.g.f. is a polynomial. For D odd, the h.g.f. is contiguous[21], i.e.,

$$F_{b-1} = \tfrac{1}{2} + \tfrac{1}{2}(1-t)F_b \tag{22}$$

where $F_b = F(1,b;3/2;t)$. Hence, using the known form for $F_{\frac{1}{2}}$, one can generate all others readily. Now since $F_{\frac{1}{2}}$ contains a logarithmic singularity (see 21), all odd-dimensioned low-k susceptibility contains the same singularity.

We next consider $F(1,\frac{1}{2};1+\frac{1}{2}D;t)$. For D even (including 0), $c = 1,2,3,...$ and for D odd, $c = 3/2, 5/2, 7/2,..$ In both cases the h.g.f. is again contiguous,

$$F_{c+1} = 3/2t.(1 - (1-t)F_c) \tag{23}$$

where $F_c = F(1,\frac{1}{2};c;t)$. Hence, now there are two "seeds," F_1 and $F_{3/2}$, where

$$F_1 = F(1,\tfrac{1}{2};1;t) = (1-t)^{-\frac{1}{2}} \tag{24}$$

and $F_{3/2}$ is already given (see 21). Thus all the even-dimensioned high k susceptibility has a square root singularity, while the odd-dimensioned high k susceptibility has a logarithmic singularity.

Shown in Table I are analytic representations of the susceptibility for $D = 0 - 6$ in the high and low k regions obtained by the relationships of the contiguous h.g.f. Even and odd dimensional cases are grouped separately to emphasize their distinctive singular behavior. These results for $D = 1 - 4$ are in agreement with the previously established results (eqs. 6 and 10). The agreement for D = 1 and 2 is interesting in view of the original restriction imposed on eq. (11), i.e., $D \geq 3$. Evidently, the dimensional regularization techniques used here have ultimately removed the restriction. Illustrated in Fig. 1 is the susceptibility vs. wave vector for a few low integer values of D.

VI. BEHAVIOR NEAR $2k_F$

The h.g.f. $F(a,b;c;t)$ is absolutely convergent on $|t| = 1$ if $Re(c-a-b) > 0$ and has the value[21]

$$F(a,b;c;1) = \Gamma(c).\Gamma(c-a-b)/\Gamma(c-a).\Gamma(c-b). \tag{27}$$

If applied to the high and low k sides, we find that

$$\tilde{\chi}_1(z=1,D) = \tilde{\chi}_2(z^- = 1,D) = (D-1)^{-1}, \quad D > 1. \tag{28}$$

Thus, the susceptibility is continuous at $z = 1$ ($k = 2k_F$) except when $D = 1$.

The slope at the boundary can be evaluated by using

$$\frac{\partial}{\partial t} F(a,b;c;t) = \frac{a.b}{c} F(a+1,b+1;c+1;t) \tag{29}$$

and (27) provided now that $Re(c-a-b-1) > 0$. We obtain

$$\frac{\partial}{\partial z} \tilde{\chi}_1(z=1,D) = \frac{\partial}{\partial z} \tilde{\chi}_2 \ (z = 1,D) = -(D-2)/(D-1)(D-3), \quad D > 3. \tag{30}$$

Similarly, we obtain

$$\left(\frac{\partial}{\partial z}\right)^2 \tilde{\chi}(z=1,D) = 2(D-2)(D-4)/(D-1)(D-3)(D-5), \quad D > 5. \tag{31}$$

Thus, where convergent, we see that the high and low k sides of the boundary have the same first and second derivatives. In Table II, we have given the boundary values.

We can also examine the behavior of the susceptibility along the boundary itself, that is, the $z=1$ constant line in, say, the Dz plane. From (28) we see that the behavior is simpler, e.g., $\frac{\partial}{\partial D} \tilde{\chi}(1,D) = -(D-1)^{-2}$, etc., than the behavior in the direction perpendicular to the boundary.

We shall use other properties of the h.g.f. to establish additional properties of the susceptibility at the boundary. First of all, the h.g.f. $F(a,b;c;t)$ has two branch points, one at $t = 1$ and the other at infinity if a or b is not a negative integer. Hence, except when D is an even integer on the low k side, the boundary is a line of branch points.

Also the h.g.f. $F(a,b;c;t)$ is defined by a power series (see eq. 18) for t complex when $|t| < 1$. It is certainly regular in this domain. Hence, the susceptibility is defined even for noninteger values of D. The h.g.f. is also defined by analytic continuation when $|t| \geq 1$. It suggests, therefore, that $\tilde{\chi}_2(z,D)$ may be an analytic continuation of $\tilde{\chi}_1(z,D)$ into the high k region.

It is known that when t lies in the part of the cut plane for which $|t| \geq 1$, $|\arg(-t)| < \pi$,[21]

$$F(a,b;c;t) = B(a,b,c).(-t)^{-a}F(a,1-c+a;1-b+a;t^{-1})$$
$$+ B(b,a,c).(-t)^{-b}F(b,1-c+b;1-a+b;t^{-1}), \qquad (32)$$

where

$$B(a,b,c) = \Gamma(c).\Gamma(b-a)/\Gamma(b).\Gamma(c-a). \qquad (32a)$$

Hence, $F(a,b;c;t)$ when it has a meaning is a one valued analytic function, regular in the whole plane of t, cut along the real axis from $t = 1$ to ∞.

Let the domains $|z| < 1$ and $|z| \geq 1$ be denoted, respectively, by $\mathcal{D}_1$ and $\mathcal{D}_2$. By our definition $\tilde{\chi}_1$ which is analytic in $\mathcal{D}_1$ is $\tilde{\chi}$ in $\mathcal{D}_1$ and similarly $\tilde{\chi}_2$ is the analytic function $\tilde{\chi}$ in $\mathcal{D}_2$. Then by the above-stated properties of the h.g.f., $\mathcal{F} \equiv F(1,1-\frac{1}{2}D;3/2;z)$ is analytic in $\mathcal{D}_1 \cup \mathcal{D}_2$. If for Re z, Re $\mathcal{F} = \tilde{\chi}_1$ in $\mathcal{D}_1$ and Re $\mathcal{F} = \tilde{\chi}_2$ in $\mathcal{D}_2$, then $\tilde{\chi}_2$ is the analytic continuation of $\tilde{\chi}_1$ into $\mathcal{D}_2$.[22] By (32),

$$\mathcal{F} = D^{-1}z^{-1}\,F(1,\tfrac{1}{2};1+\tfrac{1}{2}D;z^{-1}) + \Gamma(3/2)\Gamma(\tfrac{1}{2}D)/\Gamma(\tfrac{1}{2}+\tfrac{1}{2}D).(-z)^{-1+\frac{1}{2}D}(1 - z^{-1})^{-\frac{1}{2}+\frac{1}{2}D}. \qquad (33)$$

Consider $\mathcal{F}$ when D is an odd integer first. It is sufficient to take $D = 1$ since $\mathcal{F}$ of other odd integers can be generated from it. Then, for $D = 1$,

$$\mathcal{F} = \tfrac{1}{2}z^{-\frac{1}{2}}\ln(1 + z^{-\frac{1}{2}})/(1 - z^{-\frac{1}{2}}) + \tfrac{1}{2}(-z)^{-\frac{1}{2}}. \qquad (34)$$

In the domain $\mathcal{D}_1$ the above logarithmic argument is negative. It can be resolved into real and imaginary parts, the latter of which cancels the second term of (34) exactly, leaving

$$\mathcal{F} = \tfrac{1}{2}z^{-\frac{1}{2}}\ln(1 + z^{\frac{1}{2}})/(1 - z^{\frac{1}{2}}). \qquad (35)$$

Hence, Re $\mathcal{F} = \tilde{\chi}_1$ in $\mathcal{D}_1$. It follows directly that Re $\mathcal{F} = \tilde{\chi}_2$ in $\mathcal{D}_2$. One can similarly show that whenever D is an odd integer, $\tilde{\chi}_2$ is the analytic continuation of $\tilde{\chi}_1$ into $\mathcal{D}_2$.

We next consider F when D is an even integer. Then, $b = 1 - \frac{1}{2}D$ is either zero or a negative integer and $\tilde{\chi}_1 = F(1,b;3/2;z)$ is an entire function being a polynomial. But $\tilde{\chi}_2$ is not a polynomial. The two functions are thus not related although continuous at $z = 1$. One can, in fact, show by (33) that, for Re z, Re $F = \tilde{\chi}_1$ in $\mathcal{D}_1$, but Re $F \neq \tilde{\chi}_2$ in $\mathcal{D}_2$. With this analysis we conclude that when D is an odd integer, $\tilde{\chi}_2$ is the analytic continuation of $\tilde{\chi}_1$; but when D is an even integer, it is not. When D is not an integer, the relationship established for D odd integers is expected to hold.

VII. DISCUSSION

Our result for the susceptibility obtained as a function of z and D are embodied in Fig. 2, which gives a 3-dimensional projection of $\tilde{\chi}zD$. It is reminiscent of the PVT diagram of a homogeneous fluid. The surface represents the susceptibility that is physically accessible as z and D are varied. The shape of the surface is distinguished by an unbroken "ridge" (marked in the figure by small circles). It is a line of branch points, a $z=1$ constant line. The ridge separates the surface into two sides (high and low k). The low-k side of the surface is further subdivided by the $D=2$ constant line into an area of rising curvature and an area of falling curvature. The high k side of the surface is not further divided. Hence, the ridge is folded upward for $1 < D < 2$ and folded downward for $2 < D < \infty$.

The ridge itself shows very smooth behavior, becoming singular at one end ($D=1$) and vanishing at the other end ($D=\infty$). Other z-constant lines, e.g. $z = 0$ are less interesting. More interesting is D-constant lines which interesect the ridge. They look much like the familiar isotherms in the PV diagram (also compare Fig. 1). When D is an integer, the intersection is a point of singularity, either open as in $D = 1$ or hidden as in $D = 2$. The ridge is punctuated with these intersections throughout.

Other finer details of the susceptibility surface are possible to give. Except on the D-constant lines of <u>even</u> integers, one can move across the ridge via analytic continuation. These excepted lines are demarcated by the ridge. That is, on these excepted lines, the knowledge of one side is insufficient to describe the other side.[23] The singularity at $z = D = 1$ is weaker when approached perpendicular to the ridge than when approached along the ridge. To some extent our picture is applicable to an interacting electron gas by virtue of the RPA theories.[24]

Acknowledgments

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VIII. References

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18. See eq. 1.79e of Ref. 17.

19. The upper limits on the sums (14 and 15) at first appear troublesome if one takes $s \to 0$. A moment of reflection assures us that clearly there must be an infinite number of terms in that limit.

20. By definition the homogeneous static susceptibility $\chi(k=0) = \partial\rho/\partial\varepsilon_F$, where $\rho = 2(k_F/2\pi)^D \cdot S_D/D$, the density of electrons in D dimensions.

21. See e.g. G. Sansone and J. Gerretsen, Lectures on the Theory of Functions of a Complex Variable (Wolters-Noordhoff, Gronigen, 1969), Chapt. 16.

22. Since $\chi(z,D)$ is real for Re z, we shall add this reality requirement.

23. A similar situation occurs in the Yang-Lee theory of phase transitions in which a line of zeros of the grand partition function separates high and low temperature regions. Although continuous, pressure in one region cannot in general be determined from it in another region. See e.g. G.E. Uhlenbeck and G. Ford, Lectures in Statistical Mechanics (Am. Math. Soc., Providence, 1963) pp. 66-71.

24. In this regard, also see Y.R. Wang, M. Ashraf and A.W. Overhauser, Phys. Rev. B 30, 5580 (1984).

Captions

Table I. Analytic expressions of the susceptibility. These results are obtained by the relationships of the contiguous h.g.f. $L_1 = \ln(1 + z^{\frac{1}{2}})/(1 - z^{\frac{1}{2}})$ and $L_2 = \ln(1 + z^{-\frac{1}{2}})/(1 - z^{-\frac{1}{2}})$.

Table II. Boundary values of the susceptibility. $\tilde{\chi}'$ and $\tilde{\chi}''$ are, respectively, the first and second derivatives of $\tilde{\chi}(z,D)$ with respect to z at z = 1. These undesignated ∞'s are divergent as $(1 - z^{-1})^{-1}$ for D even and $(1 - z^{-\frac{1}{2}})^{-1}$ for D odd. In the unfilled regions, the appropriate formulas are given.

Fig. 1. The susceptibility vs wave vector at integer values of D. k in units of k_F.

Fig. 2. The susceptibility as a function of z and D. Small circles form a line of branch points. $z = (k/2)^2$.

Table I

D	$\tilde{X}_1$	$\tilde{Y}_2$
0	$z^{-\frac{1}{2}}(1 - z)^{-\frac{1}{2}} \sin^{-1} z$	$D^{-1}\, z^{-1}(1 - z^{-1})^{-\frac{1}{2}}$, $D \to 0$
2	1	$1 - (1 - z^{-1})^{\frac{1}{2}}$
4	$1 - 2z/3$	$1 - 2z/3 \cdot (1 - (1 - z^{-1})^{3/2})$
6	$1 - 4z/3 + 8z^2/15$	$1 - 4z/3 + 8z^2/15 \cdot (1 - (1 - z^{-1})^{5/2}$
..		
1	$\frac{1}{2} z^{-\frac{1}{2}}\, L_1$	$\frac{1}{2} z^{-\frac{1}{2}}\, L_2$
3	$\frac{1}{2} + \frac{1}{4} z^{-\frac{1}{2}}(1 - z) L_1$	$\frac{1}{2} + \frac{1}{4} z^{-\frac{1}{2}}(1 - z) L_2$
5.	$5/8 - 3z/8 + 3/16 z^{-\frac{1}{2}}(1 - z)^2 L_1$	$5/8 - 3z/8 + 3/16 z^{-\frac{1}{2}}(1 - z)^2 L_2$

Table II

D	$\tilde{X}_1$	$\tilde{X}_2$	$\tilde{X}_1'$	$\tilde{X}_2'$	$\tilde{X}_1''$	$\tilde{X}_2''$
1	∞(log)	∞(log)	∞	$-\infty$	∞	∞
2	1	1	0	$-\infty$	0	∞
3	½	½	$-\infty$(log)	$-\infty$(log)	∞	∞
4	1/3	1/3	-3/2	-3/2	0	∞
5			-3/8	-3/8	∞(log)	∞(log)
6					16/15	16/15
.	$\frac{1}{(D-1)}$		$\frac{-(D-2)}{(D-1)(D-3)}$			
.					$\frac{2(D-2)(D-4)}{(D-1)(D-3)(D-5)}$	
.						
∞	0	0	0	0	0	0

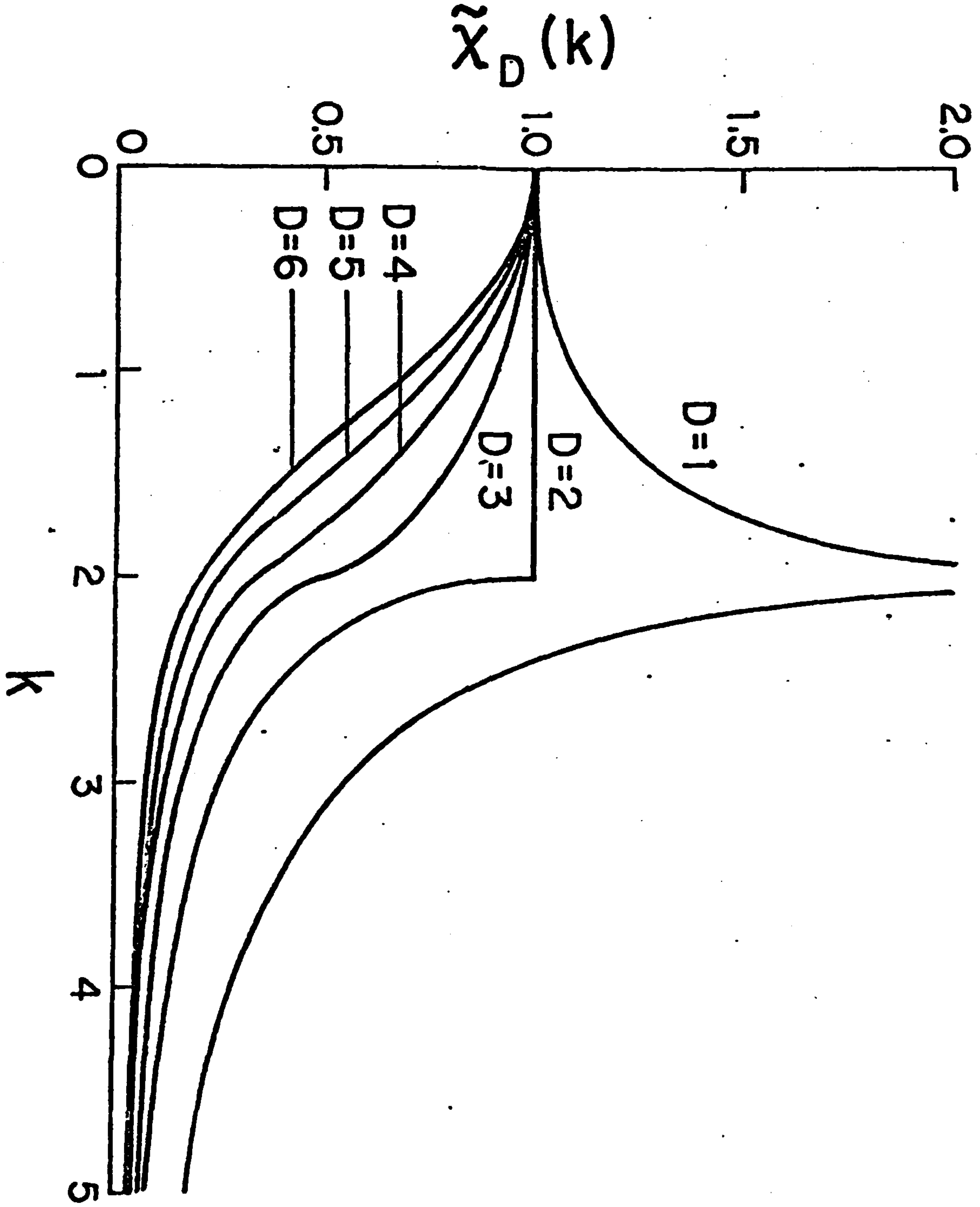
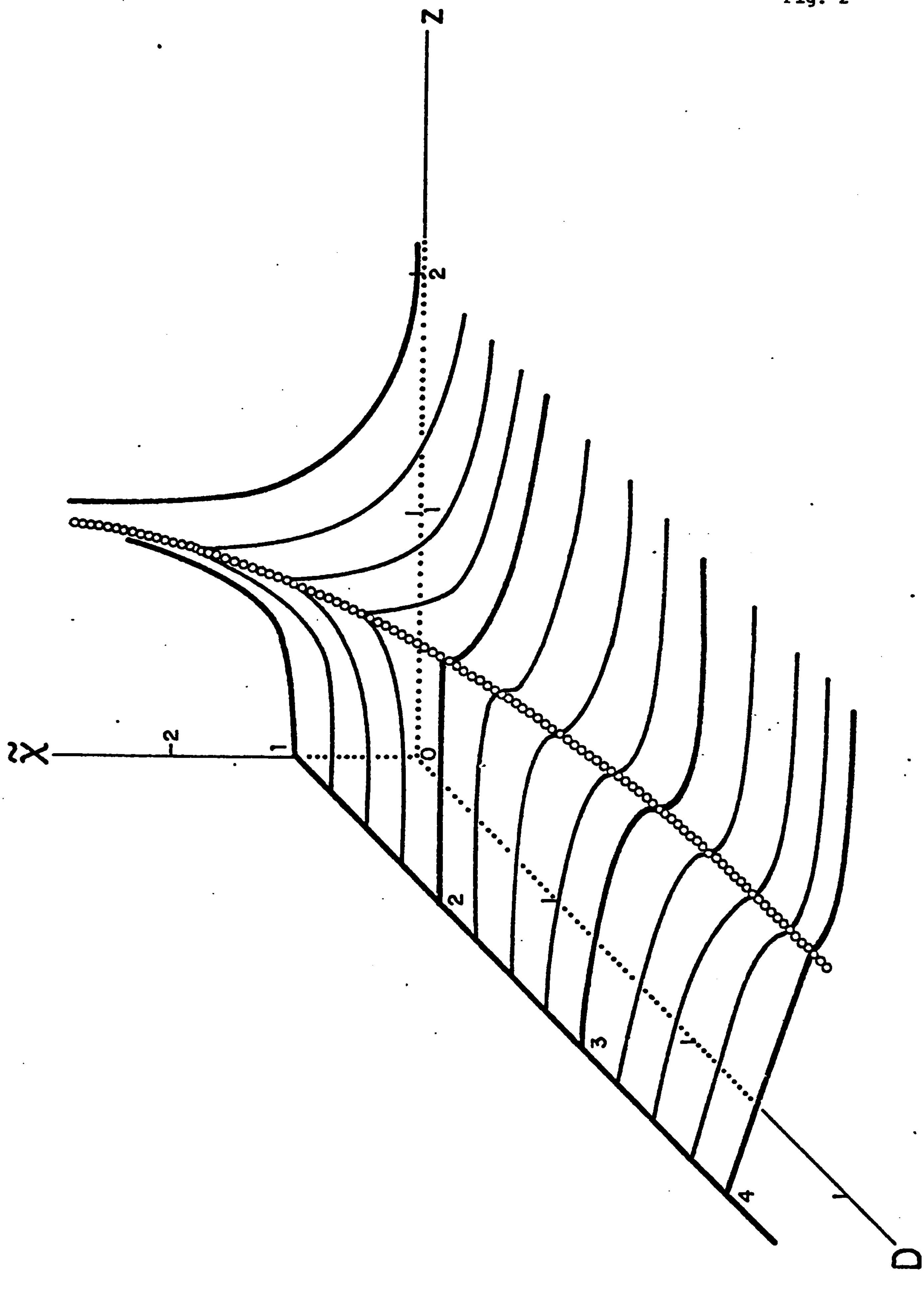

$\tilde{\chi}_D(k)$
0
0.5
1.0
1.5
2.0
0
1
2
3
4
5
k
D=1
D=2
D=3
D=4
D=5
D=6

Fig. 2



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